Impulse Response & Solutions to DE

Introduction.

When we first began with Euler's method we developed two "discretizations" of 1^{st} order DE

$$y' = f(x)$$

solve for y_k in Euler's method

$$\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix} = \begin{bmatrix}1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix}\begin{bmatrix} f_0\\ f_1\\ \vdots\end{bmatrix}\cdot h$$

$$\left(y = \int f \, dx\right)$$

solve for f_k in Euler's method

$$\frac{1}{h}\cdot\begin{bmatrix}1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix} = \begin{bmatrix}f_0\\ f_1\\ \vdots\end{bmatrix}$$

$$\left(\frac{d}{dx} y = f\right)$$

Previously we generalized the second method to 2^{nd} order DE.
Generalizing first method leads to <u>Discrete Impulse Response</u>

Idea: k^{th} column of $\begin{bmatrix}1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix}$

is solution to $\begin{bmatrix} y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix} = \begin{bmatrix}1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix}\begin{bmatrix} 0\\ \vdots\\ 1/h\\ 0\\ \vdots\end{bmatrix}\cdot h$ position k

↳ solution to $y' = \delta(x - x_k)$

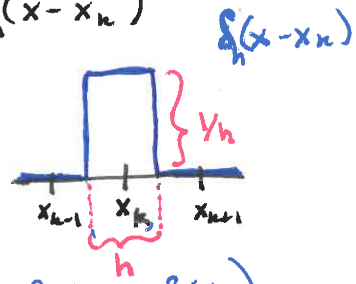
Discrete Impulse Basis

Given an h and interval $[a,b]$ for discretizing

Write $\underline{\delta}^{(k)}$ for the discrete impulse function with impulse at x_k

$$\underline{\delta}^{(k)} = \text{Discretization of } \delta_h(x - x_k)$$

$$\underline{\delta}^{(1)} = \frac{1}{h}\begin{bmatrix}1\\0\\0\\ \vdots\end{bmatrix}, \quad \underline{\delta}^{(2)} = \begin{bmatrix}0\\1\\0\\ \vdots\end{bmatrix}, \text{ etc}$$



$$\left(\lim_{h\to 0} \delta_h(x) = \delta(x)\right)$$

Just as we write vectors as sums of $\underline{i}, \underline{j}, \underline{k}$ in MAT 120

$$\langle 2, -3, 5\rangle = 2\underline{i} - 3\underline{j} + 5\underline{k}$$

we can write discrete functions as sums of discrete impulses

EX:

$$\begin{bmatrix}2\\-3\\5\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= 2\cdot h \frac{1}{h}\begin{bmatrix}1\\0\\0\end{bmatrix} - 3\cdot h \frac{1}{h}\begin{bmatrix}0\\1\\0\end{bmatrix} + 5\cdot h \frac{1}{h}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

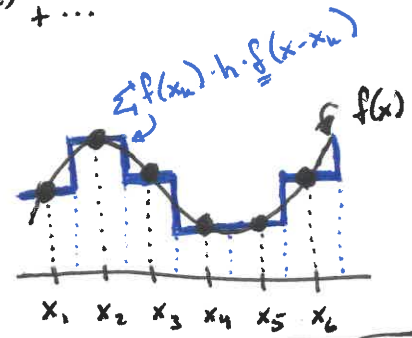
$$= 2h\,\underline{\delta}^{(1)} - 3h\,\underline{\delta}^{(2)} + 5h\,\underline{\delta}^{(3)}$$

Formula: $\begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix} = f_1 \cdot h \cdot \underline{\underline{\delta}}^{(1)} + f_2 \cdot h \cdot \underline{\underline{\delta}}^{(2)} + \cdots$

i.e. $\underline{\underline{f}} = \sum_k' f_k \cdot h \cdot \underline{\underline{\delta}}^{(k)}$

Discretization of $f(x)$

Discretization of $\sum_k' f(x_k) \cdot h \cdot \delta(x - x_k)$



This formula should look familiar — something similar appears in differential equations (end of Laplace chapter)

(MAT 210) $\underline{\underline{f}} = \sum_k' f_k \cdot h \cdot \underline{\underline{\delta}}^{(k)}$

$= \sum_k' f(x_k) \cdot \underline{\underline{\delta}}(x - x_k) \cdot h$

(MAT 219) $f(x) = \int_{\tau=0}^{\tau=\infty} f(\tau)\, \delta(x - \tau)\, d\tau$

$= f(x) * \delta(x)$

("f is the convolution of itself with impulse at 0")

Notation: Given a differential equation, $y'' + p(x)y' + q(x)y = f(x)$

write $\underline{\underline{y}}^{(k)}$ for the solution to the discretization of $y'' + p(x)y' + q(x)y = \delta(x - x_k)$

$\underline{\underline{y}}^{(k)}$ is the "k^{th} discrete impulse response"

Note: If $M \underline{\underline{y}}^{(1)} = \underline{\underline{\delta}}^{(1)}$ & $M \underline{\underline{y}}^{(2)} = \underline{\underline{\delta}}^{(2)}$

then $M(a\, \underline{\underline{y}}^{(1)} + b\, \underline{\underline{y}}^{(2)}) = a\, \underline{\underline{\delta}}^{(1)} + b\, \underline{\underline{\delta}}^{(2)}$

In particular $M\left(\sum_k' f_k \cdot h\, \underline{\underline{y}}^{(k)}\right) = \sum_k' f_k \cdot h\, \underline{\underline{\delta}}^{(k)} = \underline{\underline{f}}$

Result: The discretization of a linear DE

$y'' + p(x)y' + q(x)y = f(x)$

has

$\underline{\underline{f}} = f_1 \cdot h\, \underline{\underline{\delta}}^{(1)} + f_2 \cdot h\, \underline{\underline{\delta}}^{(2)} + \cdots$

with solution

$\underline{\underline{y}} = f_1 \cdot h\, \underline{\underline{y}}^{(1)} + f_2 \cdot h \cdot \underline{\underline{y}}^{(2)} + \cdots$

Replace impulse by impulse response!

$= h \begin{bmatrix} | & | & \\ y^{(1)} & y^{(2)} & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \end{bmatrix}$

Matrix whose columns are $\underline{\underline{y}}^{(k)}$

$y = f * y^{(k)}$ f convolved with impulse response

Note: If a differential equation discretizes to $M \underline{\underline{y}} = \underline{\underline{f}}$

Then $h \cdot \underline{\underline{y}}^{(k)}$ is the k^{th} column of M^{-1}!

EX Write the discretization of $f(x)=x^2$ on $[0,2]$ with $h=\frac{1}{2}$ as a sum of impulses.

x	$f(x)=x^2$
$x_0=0$	$f_0=0$
$x_1=\frac{1}{2}$	$f_1=\frac{1}{4}$
$x_2=1$	$f_2=1$
$x_3=\frac{3}{2}$	$f_3=\frac{9}{4}$
$x_4=2$	$f_4=4$

$$\underline{f}=0\cdot\tfrac{1}{2}\,\underline{\delta}^{(0)}+\tfrac{1}{4}\cdot\tfrac{1}{2}\,\underline{\delta}^{(1)}+1\cdot\tfrac{1}{2}\,\underline{\delta}^{(2)}+\tfrac{9}{4}\cdot\tfrac{1}{2}\,\underline{\delta}^{(3)}+4\cdot\tfrac{1}{2}\,\underline{\delta}^{(4)}$$

EX The differential equation $y''+py'+qy=f$ on $[-1,1]$ with $h=\frac{1}{2}$ has discrete impulse responses

$$\underline{y}^{(1)}=\begin{bmatrix}1\\2\\0\end{bmatrix}\quad \underline{y}^{(2)}=\begin{bmatrix}2\\0\\1\end{bmatrix}\quad \underline{y}^{(3)}=\begin{bmatrix}0\\1\\1\end{bmatrix}$$

What is the solution to $y''+py'+qy=4x+1$

x	$f(x)=4x+1$
$x_0=-1$	$f_0=-3$
$x_1=-\frac{1}{2}$	$f_1=-1$
$x_2=0$	$f_2=1$
$x_3=\frac{1}{2}$	$f_3=3$
$x_4=1$	$f_4=5$

$$\underline{f}=-\tfrac{1}{2}\,\underline{\delta}^{(1)}+\tfrac{1}{2}\,\underline{\delta}^{(2)}+\tfrac{3}{2}\,\underline{\delta}^{(3)}$$

$$\Downarrow$$

$$\underline{y}=-\tfrac{1}{2}\,\underline{y}^{(1)}+\tfrac{1}{2}\,\underline{y}^{(2)}+\tfrac{3}{2}\,\underline{y}^{(3)}$$

$$=-\tfrac{1}{2}\begin{bmatrix}1\\2\\0\end{bmatrix}+\tfrac{1}{2}\begin{bmatrix}2\\0\\1\end{bmatrix}+\tfrac{3}{2}\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$=\begin{bmatrix}1/2\\1/2\\2\end{bmatrix}$$

Note: Simple differential equations usually have ugly impulse responses

EX $-y''=f$ with $y(-1)=0$ & $y(1)=0$ using $h=\frac{1}{2}$ has discrete impulse responses

$$\underline{y}^{(1)}=\frac{1}{8}\begin{bmatrix}3\\2\\1\end{bmatrix}\quad \underline{y}^{(2)}=\frac{1}{8}\begin{bmatrix}2\\4\\2\end{bmatrix}\quad \underline{y}^{(3)}=\frac{1}{8}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

What is the solution to $-y''=4x^2$

x	$f(x)=4x^2$
$x_0=-1$	$f_0=4$
$x_1=-\frac{1}{2}$	$f_1=1$
$x_2=0$	$f_2=0$
$x_3=\frac{1}{2}$	$f_3=1$
$x_4=1$	$f_4=4$

$$\underline{f}=\tfrac{1}{2}\,\underline{\delta}^{(1)}+0\,\underline{\delta}^{(2)}+\tfrac{1}{2}\,\underline{\delta}^{(3)}$$

$$\Downarrow$$

$$\underline{y}=\tfrac{1}{2}\,\underline{y}^{(1)}+0\underline{s}+\tfrac{1}{2}\,\underline{y}^{(3)}$$

$$=\tfrac{1}{2}\cdot\tfrac{1}{8}\begin{bmatrix}3\\2\\1\end{bmatrix}+\tfrac{1}{2}\cdot\tfrac{1}{8}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$=\tfrac{1}{16}\begin{bmatrix}4\\4\\4\end{bmatrix}$$

Change to $y(0)=0$ $\quad y(1)=0$ $\quad h=1/5$

$$\underline{y}^{(1)}=\tfrac{1}{5}\cdot\tfrac{1}{5}\begin{bmatrix}4\\3\\2\\1\end{bmatrix}\qquad \underline{y}^{(2)}=\tfrac{1}{5}\cdot\tfrac{1}{5}\begin{bmatrix}3\cdot1\\3\cdot2\\2\cdot2\\1\cdot2\end{bmatrix}=\tfrac{1}{25}\begin{bmatrix}3\\6\\4\\2\end{bmatrix}$$

$$\underline{y}^{(3)}=\tfrac{1}{25}\begin{bmatrix}2\cdot1\\2\cdot2\\2\cdot3\\1\cdot3\end{bmatrix}=\tfrac{1}{25}\begin{bmatrix}2\\4\\6\\3\end{bmatrix}\qquad \underline{y}^{(4)}=\tfrac{1}{25}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

Extra Material (Impulse Response)

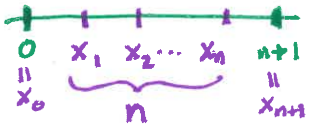
Sometimes we can figure out impulse response *without* actually computing the solution to equations!

EX: $-y'' = f(x)$ on $[0, n+1]$ with $h = 1$

so that there are n "middle" x's

$y(0) = 0$ & $y'(n+1) = 0$

"fixed" – "free"

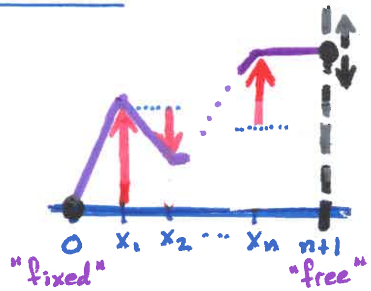


Discretization is:

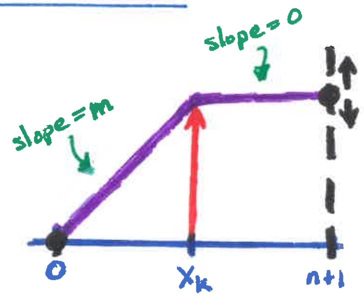
$$-\begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ 0 & & 1 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \to \text{Impulse response?}$$

This fixed-free DE describes position of an elastic string with one end attached at 0 and the other end able to slide up and down.

Forces of $f(x_k)$ are applied at each position x_k.



If $f(x) = \delta(x - x_k)$ then the picture is very simple: The string will follow two straight lines.



→ Compute upwards slope, m, using DE

Computing m:

Since $y'' = -\delta(x - x_k)$, net change theorem (MAT 119) says

$$\begin{pmatrix}\text{change in} \\ \text{slope at } x_k\end{pmatrix} = \begin{pmatrix}\text{slope after} \\ x_k\end{pmatrix} - \begin{pmatrix}\text{slope before} \\ x_k\end{pmatrix}$$

$$= y'(x_k + \varepsilon) - y'(x_k - \varepsilon)$$

$$= \int_{x_k - \varepsilon}^{x_k + \varepsilon} y''\, dx$$

$$= \int_{x_k - \varepsilon}^{x_k + \varepsilon} -\delta(x - x_k)\, dx = -1$$

($\int \text{Impulse} = 1$)

$\Rightarrow 0 - m = -1$ so $m = 1$.

Thus $\underline{y}^{(k)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ k \\ k \\ \vdots \\ k \end{bmatrix}$ $\begin{matrix} \}\text{Increase by } 1 \\ \}\text{Constant at } k \end{matrix}$

For example, if $n = 4$ then impulse responses are:

$$y^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad y^{(2)} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} \quad y^{(3)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} \quad y^{(4)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Also these combine to give the inverse matrix

$$-\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xleftrightarrow{\text{inverse}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

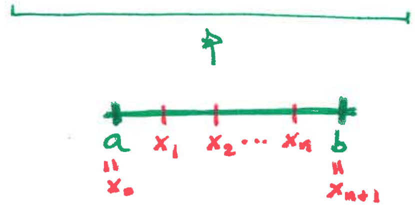
(It is easy to check this...)

It is not hard to show that in the general case, everything is multiplied by h

Impulse response for "fixed-free"

$-y'' = f(x)$ on $[a,b]$ with $h = \frac{b-a}{n+1}$

$y(a)=0$ & $y'(b)=0$



$$y^{(k)} = h\begin{bmatrix}1\\2\\\vdots\\k\\k\\\vdots\\k\end{bmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}1\\2\\\vdots\\k\end{matrix}}\right\}\text{increase by } 1\\ \left.\vphantom{\begin{matrix}k\\\vdots\\k\end{matrix}}\right\}\text{const. at } k\end{matrix}$$

Also the reverse problem has flipped impulse response:

Impulse response for "free-fixed"

$-y'' = f(x)$ on $[a,b]$ with $h = \frac{b-a}{n+1}$

$y'(a)=0$ & $y(b)=0$

$$y^{(n-k+1)} = h\begin{bmatrix}k\\\vdots\\k\\k\\\vdots\\2\\1\end{bmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}k\\\vdots\\k\end{matrix}}\right\}\text{constant at } k\\ \uparrow\text{increase by } 1\end{matrix} \qquad y^{(k)} = h\begin{bmatrix}n+1-k\\\vdots\\n+1-k\\\vdots\\2\\1\end{bmatrix}\begin{matrix}\text{const.}\\ \uparrow\text{increase}\end{matrix}$$

EX If $n=4$ & $h=\frac{1}{2}$ (e.g. on $[0,\frac{5}{2}]$ with $h=\frac{1}{2}$)

$$y^{(1)} = \tfrac{1}{2}\begin{bmatrix}4\\3\\2\\1\end{bmatrix} \quad y^{(2)} = \tfrac{1}{2}\begin{bmatrix}3\\3\\2\\1\end{bmatrix} \quad y^{(3)} = \tfrac{1}{2}\begin{bmatrix}2\\2\\2\\1\end{bmatrix} \quad y^{(4)} = \tfrac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

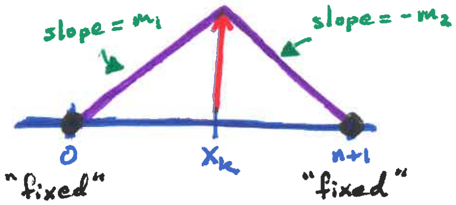
The "fixed-fixed" case is slightly trickier

EX Impulse response for

$-y'' = f(x)$ on $[0, n+1]$ with $h=1$

$y(0)=0$ & $y(n+1)=0$

Answer should be



As before, $\begin{pmatrix}\text{change in}\\\text{slope at } x_k\end{pmatrix} = -1$

$\quad\quad\quad\quad \underset{\displaystyle -m_2 - m_1}{\shortparallel}$

So $y^{(k)}$ should be $\frac{1}{m_1+m_2}\begin{bmatrix}1 \times m_1\\2\times m_1\\\vdots\\m_2\times m_1\\\vdots\\m_2\times 2\\m_2\times 1\end{bmatrix}$ } increase with slope $= m_1$ ← position k } decrease with slope $= m_2$

Divide by (m_1+m_2) so that change in slope $= 1$.

$\Longrightarrow m_2 = k$ & $m_1 = n+1-k$

(so $m_1+m_2 = n+1$)

EX: If $n=4$ then impulse responses are

$$y^{(1)} = \frac{1}{5}\begin{bmatrix}\boxed{1\cdot 4}\\1\cdot 3\\1\cdot 2\\1\cdot 1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}4\\3\\2\\1\end{bmatrix} \qquad y^{(2)} = \frac{1}{5}\begin{bmatrix}1\cdot 3\\\boxed{2\cdot 3}\\2\cdot 2\\2\cdot 1\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3\\6\\4\\2\end{bmatrix} \quad \text{etc.}$$

Impulse response for "fixed-fixed"

$-y'' = f(x)$ on $[a,b]$ with $h = \frac{b-a}{n+1}$

$y(a)=0$ & $y(b)=0$

$$y^{(1)} = h\cdot\frac{1}{n+1}\begin{bmatrix}\boxed{1\cdot n}\\ 1\cdot(n-1)\\ \vdots\\ 1\cdot 2\\ 1\cdot 1\end{bmatrix} \qquad y^{(2)} = h\cdot\frac{1}{n+1}\begin{bmatrix}1\cdot(n-1)\\ \boxed{2\cdot(n-1)}\\ 2\cdot(n-2)\\ \vdots\\ 2\cdot 2\\ 2\cdot 1\end{bmatrix} \qquad \text{etc}$$

$$y^{(k)} = h\cdot\frac{1}{n+1}\begin{bmatrix}1\cdot(n+1-k)\\ \vdots\\ \boxed{k\cdot(n+1-k)}\\ \vdots\\ k\cdot 1\end{bmatrix}$$

Ex If $n=5$ & $h=1/2$ (e.g. on $[0,3]$ with $h=1/2$)

$$y^{(1)} = \frac{1}{2}\cdot\frac{1}{6}\begin{bmatrix}\boxed{1\cdot5}\\1\cdot4\\1\cdot3\\1\cdot2\\1\cdot1\end{bmatrix} = \frac{1}{12}\begin{bmatrix}5\\4\\3\\2\\1\end{bmatrix} \qquad y^{(2)} = \frac{1}{2}\cdot\frac{1}{6}\begin{bmatrix}1\cdot4\\\boxed{2\cdot4}\\2\cdot3\\2\cdot2\\2\cdot1\end{bmatrix} = \frac{1}{12}\begin{bmatrix}4\\8\\6\\4\\2\end{bmatrix}$$

$$y^{(3)} = \frac{1}{2}\cdot\frac{1}{6}\begin{bmatrix}1\cdot3\\2\cdot3\\\boxed{3\cdot3}\\3\cdot2\\3\cdot1\end{bmatrix} = \frac{1}{12}\begin{bmatrix}3\\6\\9\\6\\3\end{bmatrix} \qquad y^{(4)} = \frac{1}{2}\cdot\frac{1}{6}\begin{bmatrix}1\cdot2\\2\cdot2\\3\cdot2\\\boxed{4\cdot2}\\4\cdot1\end{bmatrix} = \frac{1}{12}\begin{bmatrix}2\\4\\6\\8\\4\end{bmatrix}$$

$$y^{(5)} = \frac{1}{2}\cdot\frac{1}{6}\begin{bmatrix}1\cdot1\\2\cdot1\\3\cdot1\\4\cdot1\\\boxed{5\cdot1}\end{bmatrix} = \frac{1}{12}\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}$$

Note: These also combine to give the inverse

$$-\begin{bmatrix}-2&1&0&0\\1&-2&1&0\\0&1&-2&1\\0&0&1&-2\end{bmatrix} \xleftrightarrow{\text{inverse}} \frac{1}{5}\begin{bmatrix}4&3&2&1\\3&6&4&2\\2&4&6&3\\1&2&3&4\end{bmatrix}$$

$y^{(1)}$ $y^{(2)}$ $y^{(3)}$

$$-\begin{bmatrix}-2&1&0&0&0\\1&-2&1&0&0\\0&1&-2&1&0\\0&0&1&-2&1\\0&0&0&1&-2\end{bmatrix} \xleftrightarrow{\text{inverse}} \frac{1}{6}\begin{bmatrix}5&4&3&2&1\\4&8&6&4&2\\3&6&9&6&3\\2&4&6&8&4\\1&2&3&4&5\end{bmatrix}$$

[Of course, there are many other differential equations... and impulse response vectors could be all kinds of crazy things in general...]